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Energy–momentum conservation in pre-metric electrodynamics with magnetic charges

Gerald Kaiser

Center for Signals and Waves, Austin, TX, USA

E-mail: kaiser@wavelets.com

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Abstract
A necessary and sufficient condition for energy-momentum conservation is proved within a topological, pre-metric approach to classical electrodynamics including magnetic as well as electric charges. The extended Lorentz force, consisting of mutual actions by $F \sim (E, B)$ on the electric current and $G \sim (H, D)$ on the magnetic current, can be derived from an energy-momentum 'potential' if and only if the constitutive relation $G = G(F)$ satisfies a certain vanishing condition. The electric-magnetic reciprocity introduced by Hehl and Obukhov is shown to define a one-parameter family $\circledast_z$ of complex structures on the product space of 2-form pairs (F, G), independent of any spacetime metric, which reduces to the product of two Hodge star operators once a Lorentzian metric is introduced. In contrast to a recent claim made in the literature, it does *not* define a complex structure on the space of 2-forms itself.

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1. Pre-metric electrodynamics with magnetic and electric charges

In a recent monograph [HO3], Hehl and Obukhov gave an axiomatic formulation of *pre-metric* classical electrodynamics. In that approach, the initial assumptions on spacetime are minimal: it is a differentiable 4-manifold M foliated by a 'topological time' parameter σ. Maxwell's equations are expressed in terms of the *field strength* 2-form $F \sim (E, B)$, the *field excitation* 2-form $G \sim (H, D)$ and the electric current 3-form $J \sim (j, \rho)$ by

$$\mathrm{d}F = 0 \qquad F = B + E \wedge \mathrm{d}\sigma \tag{1}$$

$$\mathrm{d}G = J \qquad G = D - H \wedge \mathrm{d}\sigma \qquad J = \rho - j \wedge \mathrm{d}\sigma \tag{2}$$

where (E, H) are 1-forms, (B, D, j) are 2-forms and ρ is a 3-from, all spatial (pullbacks from the leaves). G and J are *twisted* forms, variously called *densities* [B85, p 183], *pseudoforms*

[F1] or *odd* forms [R84], while F is untwisted. Additional spacetime structures, including light cones and other elements of Lorentzian spacetime, emerge only after further conditions are imposed via *constitutive relations* defining the medium in which the fields propagate.

The purpose of this paper is to show that the magnetic flux conservation axiom in [HO3], represented above by the first equation (1), is unnecessary. To see this, note first that the only fundamental use made of $\mathrm{d}F = 0$ is in establishing the condition for energy-momentum conservation in [HO3, section B.5]. I will prove this condition *without* assuming $\mathrm{d}F = 0$ in the more general setting of n spacetime dimensions and fields represented by a p-form F and an $(n-p)$-form G. We assume that G and J are twisted, though this will play no role in the proof. For $n = 4$ and $p = 2$, the conservation of magnetic flux is replaced by the conservation of magnetic charge, whose current is represented by the 3-form $\mathrm{d}F$.

Magnetic charges have been discussed extensively by Schwinger [S75], and conserved energy-momentum tensors have been constructed in the presence of magnetic currents by Rund [R77] and Moulin [M1]. But to the best of my knowledge, the construction given here is the first that does not presuppose a spacetime metric or constitutive relations. I will establish a necessary and sufficient condition for electromagnetic energy-momentum conservation in the above setting. The choice of a Lorentzian metric then provides a sufficient, but not necessary, condition.

Given a vector field u on M, denote the contraction of F by u and its Lie derivation along u by

$$uF \equiv u\rfloor F \qquad \mathcal{L}_u F = \mathrm{d}(uF) + u\,\mathrm{d}F.$$

Since the $(n+1)$-forms $F \wedge \mathrm{d}G$ and $\mathrm{d}F \wedge G$ must vanish, we have

$$\begin{aligned} u(F\wedge \mathrm{d}G) &= uF\wedge \mathrm{d}G + \hat{p}F\wedge u\,\mathrm{d}G = 0 \qquad \text{where} \quad \hat{p}\equiv(-1)^p \\ u(\mathrm{d}F\wedge G) &= u\,\mathrm{d}F\wedge G - \hat{p}\,\mathrm{d}F\wedge uG = 0. \end{aligned} \tag{3}$$

Therefore,

$$\begin{aligned} \mathrm{d}(F\wedge uG) &= \mathrm{d}F\wedge uG + \hat{p}F\wedge \mathrm{d}(uG) \\ &= \mathrm{d}F\wedge uG + \hat{p}F\wedge(\mathcal{L}_u G - u\,\mathrm{d}G) \\ &= \hat{p}F\wedge\mathcal{L}_u G + \mathrm{d}F\wedge uG + uF\wedge \mathrm{d}G \\ &= \hat{p}F\wedge\mathcal{L}_u G + f_u \end{aligned} \tag{4}$$

where

$$f_u \equiv \mathrm{d}F\wedge uG + uF\wedge \mathrm{d}G. \tag{5}$$

Similarly,

$$\begin{aligned} \hat{p}\,\mathrm{d}(uF\wedge G) &= \hat{p}(\mathrm{d}(uF)\wedge G - \hat{p}uF\wedge \mathrm{d}G) \\ &= \hat{p}(\mathcal{L}_u F - u\,\mathrm{d}F)\wedge G - uF\wedge \mathrm{d}G \\ &= \hat{p}\mathcal{L}_u F\wedge G - \mathrm{d}F\wedge uG - uF\wedge \mathrm{d}G \\ &= \hat{p}\mathcal{L}_u F\wedge G - f_u. \end{aligned} \tag{6}$$

Taking the sum of (4) and (6) gives

$$\mathrm{d}(uF\wedge G + \hat{p}F\wedge uG) = \mathrm{d}u(F\wedge G) = \mathcal{L}_u F\wedge G + F\wedge\mathcal{L}_u G = \mathcal{L}_u(F\wedge G)$$

which is trivial by the definition of $\mathcal{L}_u$ since $u\,\mathrm{d}(F\wedge G) = 0$ as $F\wedge G$ is an n-form. Therefore, the entire content of (4) and (6) is in the *difference,* which we write as

$$\mathrm{d}\Sigma_u = f_u + \phi_u \tag{7}$$

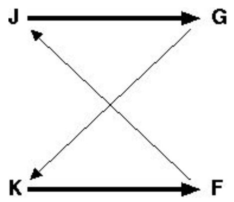

Figure 1. Reciprocity in action. The field F generated by the magentic current K acts on the electric current J, and the field G generated by J in turn acts on K.

where

$$\Sigma_u = \frac{1}{2}(F \wedge uG - \hat{p}uF \wedge G) \tag{8}$$

$$\phi_u = \frac{\hat{p}}{2}(F \wedge \mathcal{L}_u G - \mathcal{L}_u F \wedge G). \tag{9}$$

Specializing to $n = 4$ and $p = 2$, F is interpreted as the field strength and G as the excitation generated by the *electric and magnetic currents*

$$J \equiv \mathrm{d}G \qquad K \equiv \mathrm{d}F. \tag{10}$$

The 3-form Σ_u is the local *kinematical energy–momentum field density* and the 4-form

$$f_u \equiv K \wedge uG + uF \wedge J \tag{11}$$

is the *extended Lorentz force density,* reducing to the one defined in [HO3] for $K = 0$.[1] Equation (7) shows that Σ_u acts as a *potential* for f_u if and only if $\phi_u = 0$. That is, ϕ_u is the *obstruction* to energy-momentum conservation. The spacetime components of energy-momentum and force are obtained by choosing a local coordinate basis of vector fields $u = \partial_\alpha$ ($\alpha = 0, 1, 2, 3$).

Let us collect our expressions for $n = 4$ and $p = 2$ in the form

$$\begin{aligned} \Sigma_u(F, G) &= \tfrac{1}{2}(F \wedge uG - G \wedge uF) \\ f_u(F, G) &= uF \wedge \mathrm{d}G - uG \wedge \mathrm{d}F \\ \phi_u(F, G) &= \tfrac{1}{2}(F \wedge \mathcal{L}_u G - G \wedge \mathcal{L}_u F). \end{aligned} \tag{12}$$

Note that all three expressions are bilinear in (F, G) and odd under the exchange $F \leftrightarrow G$, hence invariant under the *electric-magnetic reciprocity* substitution [HO3]

$$F \to F' = zG \qquad \text{and} \qquad G \to G' = -z^{-1}F \tag{13}$$

where z is any real nonzero parameter. Since F and F' have the dimensions of *action per unit charge* while G and G' have the dimensions of *charge,* z has the dimensions of *impedance* (usually denoted in textbooks by Z). Furthermore, to keep F' twist-free and G' twisted, z must transform as a *pseudoscalar* (or scalar of *odd type,* in the language of de Rham [R84]), changing sign under orientation-reversing coordinate transformations. Reciprocity is a precursor of *electric-magnetic duality* in the absence of a metric, as explained below. Its manifestation can be seen in the subtle structure of the extended Lorentz force, which states that the field F, generated by the magnetic current K, exerts a force on the electric current J which generates G, and G in turn exerts a force on K, as depicted in figure 1.

[1] Σ_u is 'kinematical' because in the pre-metric setting there is insufficient structure to define a *dynamics*. This is somewhat analogous to a physical system described in *phase space* (a symplectic manifold) before a specific Hamiltonian is selected.

To complete the picture, recall that the Maxwell system is underdetermined and must be supplemented with constitutive relations before a solution can be contemplated. In the present context, this takes the general form [P62]

$$G = G(F) \tag{14}$$

which includes nonlinear or even nonlocal relations that may be inconsistent with the existence of a local metric but nevertheless describe electromagnetic phenomena in media. Requiring (14) to be linear, local, symmetric and to preserve electric-magnetic reciprocity invariance generates a light-cone structure on spacetime, which determines a metric up to a conformal factor [HO3, section D.6.1] and reduces (14) to

$$G = Z^{-1} * F + \alpha F$$

where $*F$ is the *Hodge dual* of F [F1] according to a reference metric in the conformal class, Z is a conformal factor (like z, a pseudoscalar field with dimensions of impedance) and α, called an *axion field,* is also pseudoscalar. If we further assume that $\alpha = 0$ and choose the metric in the class with constant $Z = Z_0$, this reduces to

$$G = Z_0^{-1} * F \tag{15}$$

which is the *Maxwell–Lorentz spacetime relation* describing electromagnetic propagation in a (possibly curved) vacuum spacetime if Z_0 is interpreted as the *vacuum impedance* $\sqrt{\mu_0/\varepsilon_0}$ in SI units. From (15) it follows that

$$2Z_0\phi_u = F \wedge \mathcal{L}_u{*F} - \mathcal{L}_u F \wedge {*F} = F \wedge {*\mathcal{L}_u F} - \mathcal{L}_u F \wedge {*F} = 0$$

hence Σ_u is a 'potential' for f_u:

$$f_u = \mathrm{d}\Sigma_u.$$

The same argument holds for general n and p, with some sign differences.

2. Pre-metric complex structure

The Hodge duality operator associated with a Lorentzian metric defines a *complex structure* on the space Ω^2 of 2-forms, i.e.,

$$* : \Omega^2 \to \Omega^2 \qquad **F = -F. \tag{16}$$

Equations (13) and (15) show that reciprocity is a precursor of Hodge duality, i.e., that

$$F' \equiv zG \sim *F \qquad G' \equiv -z^{-1}F \sim *G. \tag{17}$$

It is tempting to suppose that reciprocity actually *defines* a pre-metric complex structure on 2-forms by writing

$$\circledast F = zG \qquad \circledast G = -z^{-1}F \tag{18}$$

as was done by Hehl and Obukhov in [HO4]. However, this is *incorrect.* In (18), we must know both F and G to compute their images under $\circledast$ and there is no way of defining $\circledast H$ for an arbitrary 2-form H. Instead, reciprocity defines a complex structure on the *product space* $\Omega^2 \times \tilde{\Omega}^2$, where Ω^2 is the space of 2-forms and $\tilde{\Omega}^2$ is the space of twisted 2-forms. Namely,

$$\circledast_z(F, G) = (zG, -z^{-1}F) \quad \Rightarrow \quad \circledast_z^2(F, G) = -(F, G) \tag{19}$$

where we have included the pseudoscalar z in the notation for the operator $\circledast_z$. In fact, definition (19) descends to the *tensor product*

$$\tilde{\Omega}^{2,2} = \Omega^2 \otimes \tilde{\Omega}^2 \tag{20}$$

since the latter is invariant under transformation on $\Omega^2 \times \tilde{\Omega}^2$ of the type

$$(F, G) \mapsto (kF, k^{-1}G) \qquad k \neq 0. \tag{21}$$

The eigenvectors of $\circledast_z$ are the *self-reciprocal* pairs

$$\left(F_z^{\pm}, G_z^{\pm}\right) = (F, G) \mp \mathrm{i} \circledast_z (F, G) = (F \mp \mathrm{i}zG, G \pm \mathrm{i}z^{-1}F) \tag{22}$$

with

$$\circledast_z \left(F_z^{\pm}, G_z^{\pm}\right) = \pm\mathrm{i}\left(F_z^{\pm}, G_z^{\pm}\right). \tag{23}$$

Note that

$$F_z^{\pm} = \mp\mathrm{i}zG_z^{\pm},$$

which is not surprising since the two-dimensional *real* space spanned by the pair (F, G) (at a given spacetime point) is equivalent to a one-dimensional *complex* space. When a Lorentzian metric is chosen and the vacuum relations (15) are satisfied, the self-reciprocal pairs with $z = Z_0$ become

$$\left(F_z^{\pm}, G_z^{\pm}\right) = (F \mp \mathrm{i} * F, G \mp \mathrm{i} * G) = (F^{\pm}, G^{\pm}) \tag{24}$$

where $F^{\pm}$ and $G^{\pm}$ are the *self-dual* forms with respect to the Hodge operator of the given metric,

$$*F^{\pm} = \pm\mathrm{i}F^{\pm} \qquad *G^{\pm} = \pm\mathrm{i}G^{\pm}.$$

To summarize, we have two levels of complex structure.

- Given only a differentiable 4-manifold M, there is a one-paramater family of complex structures $\circledast_z (z \neq 0)$ on $\Omega^2 \times \tilde{\Omega}^2$ preserving the kinematical energy-momentum form Σ_u and the extended Lorentz force f_u.
- Choosing a Lorentzian metric on M and assuming the vacuum relation (15), the reciprocity operator $\circledast_z$ with $z = Z_0$ *factors* into the product of two Hodge operators defined by the metric:

$$\circledast_z (F, G) = (*F, *G). \tag{25}$$

- The class of constitutive relations equation (14) yielding the condition $\phi_u = 0$ (equivalent to the potential condition $f_u = \mathrm{d}\Sigma_u$) will in general depend on K because ϕ_u involves the exterior derivatives $\mathrm{d}F$. The specific connection between this class and the choice of K remains to be investigated[2].

3. Conclusions

It has been shown that the pre-metric formulation of classical electrodynamics given in [HO3] has a natural generalization. The conservation of magnetic flux and electric charge ($\mathrm{d}F = 0, \mathrm{d}G = J$) is replaced by the duality-symmetric conservation of magnetic and electric charge ($\mathrm{d}F = K, \mathrm{d}G = J$). The potential relation $f_u = \mathrm{d}\Sigma_u \iff \phi_u = 0$ remains true if the Lorentz force in [HO3] is replaced by its symmetric version equation (12). The symmetry of the extended theory under electric-magnetic duality is reflected by the existence of the one-parameter family of complex structures $\circledast_z$ on $\Omega^2 \times \tilde{\Omega}^2$ which factor into products of Hodge star operators equation (25) once a Lorentzian metric is assumed so that the Maxwell–Lorentz relation $G = Z_0^{-1} * F$ is satisfied.

[2] I thank the referee for raising this question.

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